Rayleigh instability of a thermal boundary layer in flow through a porous medium

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It is supposed that a heated liquid is rising very slowly through a semi-infinite porous medium towards the permeable horizontal surface, where it mixes with a layer of cool overlying fluid. In the steady state a thermal boundary layer of exponential form exists in the medium. It is shown that the layer is stable provided that the Rayleigh number for the system does not exceed a critical positive value, and that the wave-number of the critical neutral disturbance is finite. The stability properties of the layer are explained qualitatively from physical considerations.

1. Introduction

In the geothermal region of Wairakei, New Zealand, it is known that the subsurface ground water possesses a general upward convective drift, due to buoyancy induced by the high underground temperature. Since the rising ground water is cooled as it approaches the surface, where heat is removed by evaporation, radiation and movement in surface streams, an unstable state may be induced, and complicated convective motions appear in the layers near the surface. It is the purpose of this paper to investigate the conditions for instability to occur.

In an idealized case, one considers a dynamically incompressible fluid, which rises at a constant, uniform rate through a semi-infinite homogeneous porous medium, and passes through the surface to mix with a layer of fluid at constant temperature. When a steady state has been established, a thermal boundary layer of exponential form then exists below the surface.

The basic equations. Suppose that a homogeneous isotropic porous medium of porosity ϵ and permeability k is saturated with a liquid incompressible to pressure changes, the density ρ being a function of temperature T only. To allow for the dilatation and contraction of the fluid with temperature variations, it is convenient to introduce a vector $\mathbf{q}_m$ proportional to the rate of mass flow, and related to the usual volume flow vector $\mathbf{q}$ by

$$\mathbf{q}_m = \frac{1}{\rho_0}\rho\mathbf{q}, \tag{1}$$

where ρ_0 is a reference density, corresponding to a temperature T_0. Then the

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equations of continuity, motion and heat transport for the liquid in the porous medium are

$$\frac{\epsilon}{\rho_0}\frac{\partial \rho}{\partial t}+\operatorname{div}\mathbf{q}_m = 0, \tag{2}$$

$$\frac{1}{\rho_0}\operatorname{grad}P-\mathbf{g}\rho/\rho_0+\frac{1}{k}\nu\mathbf{q}_m = 0, \tag{3}$$

$$E\frac{\partial T}{\partial t}+\mathbf{q}_m.\operatorname{grad}T = \operatorname{div}(\kappa\operatorname{grad}T). \tag{4}$$

In equation (3), P is the pressure, $\nu\equiv\nu(\rho) = \mu/\rho$ is the kinematic viscosity ($\mu\equiv\mu(\rho)$ being the dynamic viscosity) and $\mathbf{g}$ is the acceleration due to gravity. Inertial terms do not appear in this equation, which is usually called Darcy's law. The conditions for the validity of equation (3) are that the Reynolds number based upon the flow through the pores should not exceed $O(1)$, and that the time scale of unsteady macroscopic motions should be very much greater than k/ν—the time scale of transient motions in a single pore.

In equation (4),

$$c\rho_0 E(\rho) = (1-\epsilon)\,c_s\rho_s+\epsilon c\rho \tag{5}$$

is the heat capacity per unit volume of saturated porous material, c being the specific heat of the liquid, and c_s and ρ_s the specific heat and the density of the solid material. Throughout this paper, the Boussinesq approximation that a linear relationship exists between the density ρ and the temperature T will be assumed, so that T can be replaced by ρ in equation (4). The diffusivity $\kappa = K/c\rho_0$, where K is the thermal conductivity of the saturated porous material, i.e. the heat flux crossing unit area in the presence of a unit temperature gradient. When mechanical dispersion is also present, κ must be replaced by a tensor quantity.

2. Formulation of the stability problem

Let the horizontal boundary of the porous medium be at $Z = 0$, the axis OZ being directed vertically upwards, and let the porous medium occupy the region $-\infty < Z \leqslant 0$. The surface is assumed to be covered by static liquid of constant temperature, which gives the boundary conditions $\rho=\rho_1=$ constant and $P=$ constant at $Z = 0$. It will be assumed also that, at $Z = -\infty$, the medium is saturated with a liquid of density ρ_0, which is rising vertically at a steady uniform mass flow rate $\rho W = \rho_0 W_m$ (cf. equation (1)). The liquid passes through the permeable boundary at $Z = 0$ and mixes with the layer of standing fluid. When this primary flow is very slow, thermal diffusion from the surface $Z = 0$ into the medium becomes important, and a thermal boundary layer appears. If the thermal diffusivity κ taken in the Z-direction can be assumed constant, the steady-state primary density distribution is, from equations (2) and (4),

$$\rho = \rho_0+(\rho_1-\rho_0)\,e^{W_m Z/\kappa} \tag{6}$$

since W_m is a constant.

Mechanical dispersion is negligible, since it can be shown that the Péclet number of the flow through the pores is of the same order of magnitude as the ratio of a typical pore diameter to the thickness κ/W_m of the thermal boundary layer. This ratio will be vanishingly small. It follows that the thermal diffusivity can be taken to be isotropic, since the medium is isotropic, so that κ is a scalar constant in equation (4).

To examine the stability of the given primary flow, one considers the effect of superimposing small perturbations p, $\mathbf{q}'_m$ and θ (say) upon the given primary values of pressure, velocity and density, respectively. The perturbed values are substituted into equations (2), (3) and (4) and, after linearizing in the usual manner, one obtains the perturbation equations

$$\frac{\epsilon}{\rho_0}\frac{\partial \theta}{\partial t} + \operatorname{div} \mathbf{q}'_m = 0, \tag{7}$$

$$\frac{1}{\rho_0}\operatorname{grad} p + \mathbf{k}\left(\frac{1}{\rho_0}g + \frac{1}{k}\frac{d\nu}{d\rho}W_m\right)\theta + \frac{1}{k}\nu\mathbf{q}'_m = 0, \tag{8}$$

$$E\frac{\partial \theta}{\partial t} + W_m\frac{\partial \theta}{\partial Z} + \frac{\partial \rho}{dZ}w_m = \kappa\nabla^2\theta, \tag{9}$$

where $\mathbf{k}$ is the unit vector in the upward (Z-) direction, and w_m is the Z-component of $\mathbf{q}'_m$. In these equations, the quantity $d\nu/d\rho$ is assumed known from the properties of the given liquid, while $d\rho/dZ$ is given by (6). The pressure perturbation p and the horizontal components of $\mathbf{q}'_m$ can be eliminated by taking the divergence of (8), using (7), to give

$$\operatorname{div}(\nu \operatorname{grad} w_m) + \frac{\epsilon}{\rho_0}\frac{\partial}{\partial Z}\left(\nu\frac{\partial \theta}{\partial t}\right) + \left(\frac{gk}{\rho_0} + \frac{d\nu}{d\rho}W_m\right)\nabla_1^2\theta = 0, \tag{10}$$

where $\nabla_1^2 \equiv \nabla^2 - \partial^2/\partial Z^2$. When w_m has been eliminated between (10) and (9), one obtains the linearized equation obeyed by the small density perturbation θ

$$\operatorname{div}\left[\nu \operatorname{grad}\left\{\left(\frac{d\rho}{dZ}\right)^{-1}\left(\kappa\nabla^2\theta - W_m\frac{\partial \theta}{\partial Z} - E\frac{\partial \theta}{\partial t}\right)\right\}\right] + \frac{\epsilon}{\rho_0}\frac{\partial}{\partial Z}\left(\nu\frac{\partial \theta}{\partial t}\right) + \left(\frac{gk}{\rho_0} + W_m\frac{d\nu}{d\rho}\right)\nabla_1^2\theta = 0. \tag{11}$$

The boundary conditions upon θ are as follows. At $Z = 0$, $\theta = 0$ and, using equations (2) and (3), one finds that the boundary condition $p = 0$ is equivalent to $\partial w_m/\partial Z = 0$ (Lapwood 1948). In terms of θ,

$$\frac{\partial}{\partial Z}\left\{\left(\frac{d\rho}{dZ}\right)^{-1}\left(\kappa\nabla^2\theta - W_m\frac{\partial \theta}{\partial Z} - E\frac{\partial \theta}{\partial t}\right)\right\} = 0$$

on $Z = 0$. When $Z \to -\infty$, it is necessary that θ and the derivatives of θ should tend to zero more rapidly than $\exp(W_m Z/\kappa)$.

It will be convenient to define the natural length unit of the system by the boundary-layer thickness κ/W_m, taking a new dimensionless variable $z = W_m Z/\kappa$. The same scale unit will be used to define dimensionless variables in the horizontal co-ordinates. Also, let $\sigma = \nu/\nu_0$, where ν_0 is the kinematic viscosity of the liquid when $\rho = \rho_0$.

From the form of the perturbation equation (11), it appears that separable solutions for the density disturbance possess the well-known cellular form of Rayleigh instability (Pellew & Southwell 1940). Let

$$\theta = F(z)\,\Phi_1 e^{\omega t}, \tag{12}$$

where Φ_1 is a solution of the equation $(\nabla_1^2 + a^2)\Phi_1 = 0$ such that the normal gradient of Φ_1 vanishes on a vertical cell boundary. The wave-number a is characteristic of the variations of θ in the horizontal plane, and is dimensionless since the length κ/W_m has been taken as the unit in the horizontal co-ordinates. The operator ∇_1^2 is expressed in terms of these non-dimensional variables.

The eigenvalue equation governing $F(z)$ follows when (12) is substituted into (11). There results

$$\{D(\sigma D) - a^2\sigma\} e^{-z}(D^2 - D - a^2 - E\Omega) F + \Delta_\rho \Omega D(\sigma F) = a^2 \lambda F, \qquad (13)$$

where $$D \equiv d/dz, \quad \Delta_\rho = \epsilon(\rho_1 - \rho_0)/\rho_0, \quad \Omega = \kappa\omega/W_m^2,$$

and $$\lambda = \frac{\rho_1 - \rho_0}{\nu_0 W_m}\left(\frac{gk}{\rho_0} + W_m \frac{d\nu}{d\rho}\right).$$

In the derivation leading to equation (13), it has been assumed that the isotropic thermal diffusivity κ is a constant, independent of temperature. Further assumptions are necesary in order to simplify the eigenvalue problem represented by (13) together with the boundary conditions upon θ. Suppose that the total density difference $\rho_1 - \rho_0$ is small compared with ρ_0, so that the ratio Δ_ρ/E is small. In (13), if the term involving Δ_ρ is neglected, the approximation is equivalent to discarding the first term in the equation of continuity (7). Also, replacing E by the constant $E_0 = E(\rho_0)$ introduces an approximation of the same order. A further consequence of the assumption that $(\rho_1 - \rho_0)/\rho_0$ is small compared with unity is that the quantity $d\nu/d\rho$ can be taken constant, equal to $(d\nu/d\rho)_0$ say, which is the value when $\rho = \rho_0$. Then λ is a constant, and will be defined as the Rayleigh number for the system.

If ν is replaced by μ/ρ in the expression for λ, there results

$$\lambda = \frac{\rho_1 - \rho_0}{\mu_0 W_m}\left\{gk + W_m\left(\frac{d\mu}{d\rho}\right)_0 - W_m \frac{\mu_0}{\rho_0}\right\}, \qquad (14)$$

where μ_0 is the dynamic viscosity when $\rho = \rho_0$.

Evidently, three physical phenomena affect the stability of the system. The first is gravitational instability of the well-known Rayleigh type. The second effect was noted by Saffman & Taylor (1958) in connexion with the instability of an interface between two immiscible liquids, moving normal to itself through a porous medium. These authors showed that, when the forcing liquid has the lower viscosity, there is a tendency for instability to appear at the interface. In the corresponding case here, the sign of the second term in the last line of (14) is positive (tending to produce instability) whenever the viscosity increases in the direction of the primary flow. It is found that no analogue exists for the third term of (14) in the problem discussed by Saffman & Taylor. In the present problem the fluid is in motion with a constant rate of mass flow proportional to W_m. However, in the presence of a stationary gradient of density in the flow direction, the fluid must dilate or contract as it moves, and the magnitude of the volume flow rate W is not a constant. Since the fluid experiences a non-uniform body force proportional to $-\mu W = -\mu W_m \rho_0/\rho$ which opposes the flow, the stability of the system is increased when the density increases in the flow direction.

Using the approximate form of equation (13), it is easy to show that small disturbances vary with time in an aperiodic manner. Suppose that F and $\bar{F}$ are complex conjugate solutions of (13) which satisfy the given boundary conditions, with associated parameters (Ω, a, λ) and $(\bar{\Omega}, a, \lambda)$, respectively. (A method of Pellew & Southwell (1940) can be used to show that the wave-number a is always real.) Then, from (13), it is found that

$$(\Omega-\bar{\Omega})\int_{-\infty}^{0} e^{-z}|F|^2\,dz = 0;$$

that is, the exponent Ω in the time factor is always real. A criterion for neutral stability can be specified by putting $\Omega = 0$ in (13), when the particular disturbance being considered becomes neutral everywhere. Although this result is approximate, it would appear that, in the exact criterion for neutral stability, Ω will differ from zero by $O(\Delta_\rho)$ at most, and the error in a calculated eigenfunction solution will be of the same order. The error in calculating a criterion for neutral stability will be $O(\Delta_\rho^2)$.

For later convenience, the functions

$$G = e^{-z}(D^2-D-a^2)F \quad \text{and} \quad H = \{D(\sigma D)-a^2\sigma\}G \tag{15}$$

are introduced. In the neutral case, G is proportional to the z-dependence of w_m, the perturbation to the vertical component of the disturbance motion, and H is proportional to F. Then equations (15) and the approximate form of (13) give

$$(D^2-D-a^2)H = a^2\lambda e^z G, \quad (\Omega = 0). \tag{16}$$

3. Solution of the problem

Series solution. Consider the case of neutral stability, and neglect viscosity variations for the present except where these occur in the Rayleigh number (14). Then $\Omega = 0$ and $\sigma = 1$ in (13). An appropriate series solution is

$$\begin{aligned} F &= \sum_{n=0}^{\infty} \lambda^n (A_n \exp c_1 z + B_n \exp c_2 z)\, e^{nz} \\ &= A_0 F_1 + B_0 F_2 \text{ say,} \end{aligned} \tag{17}$$

in which A_0 and B_0 are the two constants which are arbitrary. The coefficients A_{n+1} $(n \geqslant 0)$ are given by the recurrence relation

$$\frac{A_{n+1}}{A_n} = \frac{a^2}{\{(c_1+n)^2-a^2\}\{(c_1+n)(c_1+n+1)-a^2\}},$$

where $c_1 = 1+a$. For the relation giving B_{n+1}/B_n $(n \geqslant 0)$, it is necessary to replace c_1 by $c_2 = \frac{1}{2}+(\frac{1}{4}+a^2)^{\frac{1}{2}}$.

When the boundary conditions $F = DG = 0$ at $z = 0$ are applied to (17), the constants A_0 and B_0 are found to be non-zero only if the characteristic equation

$$\Delta(z) \equiv \begin{vmatrix} F_1 & F_2 \\ DG_1 & DG_2 \end{vmatrix} = 0 \tag{18}$$

is satisfied at $z = 0$, where use is made of the notation of (15). The solution of (18) gives a relation connecting λ and a. For a given value of a, there exists an infinity of distinct eigenvalues satisfying the problem, the lowest value of the set having the greatest physical significance.

A convenient method of solving (18) consists in forming the differential equation obeyed by $\Delta(z)$, solving the equation in series and then equating the appropriate series solution to zero for $z = 0$ (see for example, Nicholson (1917)). After F_1 and F_2 have been eliminated, it is found that Δ obeys the differential equation

$$(D-2)(D-1)\{D^2(D-1)^2-a^2(2D-1)^2\}\Delta+2a^2\lambda e^z(D+1)(2D-1)\Delta = 0. \quad (19)$$

Only one of the six solutions of (19) is relevant to the present problem, and is of the form

$$\sum_{n=0}^{\infty} C_n \lambda^n e^{(k+n)z}.$$

The recurrence relation governing the coefficients can be written

$$\frac{C_{n+1}}{C_n} = \frac{-2a^2(k+1+n)(2k-1+2n)}{(1+n)(k-1+n)(k+n)(2k+n)(2k-2a+n)(2a+1+n)},$$

where $k = a+\frac{1}{2}+(\frac{1}{4}+a^2)^{\frac{1}{2}}$ (the appropriate solution of the indicial equation) is determined by comparing the exponent of the leading term in the C_n-series with the corresponding exponent in (18). Putting $z = 0$ in the chosen solution of (19), one obtains an expansion of the characteristic equation

$$\Delta = \sum_{n=0}^{\infty} C_n\lambda^n = C_0\left(1+\frac{C_1}{C_0}\lambda\left(1+\frac{C_2}{C_1}\lambda\left(1+\dots\right.\right.\right.$$
$$= 0, \quad (20)$$

where the constant C_0 is non-zero. If a is given, (20) becomes an equation of infinite order in λ. Fortunately, the series converges rapidly for the lowest eigenvalue, and (20) may be solved by iteration.

a	λ
0·42669	$6{\cdot}95400_1$
0·42723	$6{\cdot}95396_3$
0·42850	$6{\cdot}95394_8$
0·42971	$6{\cdot}95396_3$
0·43032	$6{\cdot}95403_5$

TABLE 1. Values of the lowest root λ for values of a near the point of maximum instability.

Table 1 gives the values of the lowest root of equation (20) for five different values of a, using the first five terms of the series. When $a = 0{\cdot}4285$, λ passes through a minimum value of 6·9539—the critical point for neutral stability.

The forms of the functions F and G for a neutral disturbance, calculated from equations (17) and (15), are shown in figure 1 (*b*) and (*c*). A significant feature of these functions is that the velocity disturbance penetrates farther into the porous medium than does the density disturbance.

Solution by Chandrasekhar's method

A calculation of the form of the neutral curve for the lowest eigenvalue will now be given, using the well-known approximate method of Chandrasekhar (1954).

The relevant differential equations are given by (16) and the second equation in (15). These reduce to a suitable form after the transformations

$$K = e^{-\frac{1}{2}z}H \quad \text{and} \quad \zeta = -cz \quad (c = (4a^2+1)^{\frac{1}{2}})$$

have been carried out. Then

$$(D^2 - \tfrac{1}{4})K = b^2\lambda e^{-\zeta/2c}G \tag{21}$$

and

$$\{D(\sigma D) - b^2\sigma\}G = \frac{1}{c^2}e^{-\zeta/2c}K, \tag{22}$$

where $D \equiv d/d\zeta$ in this case and $b = a/c$.

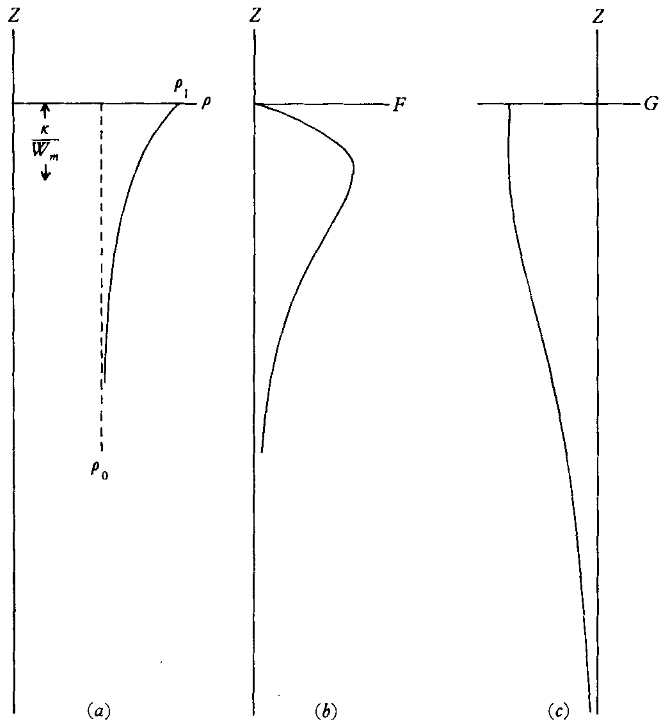

FIGURE 1. Instability of a liquid rising through a semi-infinite porous medium. (*a*) Primary density distribution; (*b*) dependence upon Z of the density perturbation at neutral stability; (*c*) dependence upon Z of the perturbation to the vertical mass flow at neutral stability.

In the right-hand side of (22), one can substitute for K the expansion in orthogonal functions

$$K(\zeta) = \zeta e^{-\frac{1}{2}\zeta} \sum_{n=0}^{\infty} A_n L_n^1(\zeta), \tag{23}$$

where the A_n are constants and the

$$L_n^1(\zeta) \equiv \frac{e^{\zeta}}{n!\,\zeta} D^n(\zeta^{n+1}e^{-\zeta})$$

are Laguerre polynomials (Morse & Feshbach 1953, p. 784). The set of Laguerre polynomials $L_n^1(\zeta)$ $(n = 0, 1, 2, \ldots)$ is complete, and the expansion (23) satisfies the boundary conditions $K = 0$ at $\zeta = 0$ and $K \to 0$ exponentially as $\zeta \to +\infty$.

Assuming for the present that $\sigma = 1$ in (22), one solves the equation for G, subject to the boundary conditions $DG = 0$ at $\zeta = 0$ and $G \to 0$ exponentially as $\zeta \to +\infty$. This solution and the expansion (23) for K are substituted into (21). Now, if (21) is multiplied through by $\zeta e^{-\frac{1}{2}\zeta} L_m^1(\zeta)$ $(m = 0, 1, 2, \ldots)$ and integrated with respect to ζ over the range $(0, \infty)$, an infinite set of equations results

$$\sum_{n=0}^{\infty} A_n \left\{ I_{mn} - \frac{2c^3}{a\lambda} \delta_{mn}(n+1)^2 \right\} = 0 \quad (m = 0, 1, 2, \ldots), \tag{24}$$

where δ_{mn} is the Kronecker delta, and

$$I_{mn} = \{I_1(Q,P) + I_1(Q,Q) + I_2(Q,P) - I_2(P,Q)\}_{mn},$$

in which

$$(I_1)_{mn}(P,Q) = (m+1)(n+1)\frac{(P-1)^m (Q-1)^n}{P^{m+2} Q^{n+2}}$$

and

$$(I_2)_{mn}(P,Q) = \frac{m+1}{n!}\left[\frac{d^{n+1}}{du^{n+1}} \frac{(P+Q-u-1)^m (u-1)^n}{(P+Q-u)^{m+2}(u-Q)}\right]_{u=0}.$$

Here $P = (c+1-2a)/2c$ and $Q = (c+1+2a)/2c$, with c defined as before. In the particular case $m = n = 0$, it is found that

$$I_{00} = \frac{1}{Q}\left\{\frac{1}{Q^3} + \frac{2}{(P+Q)^3}\left(\frac{P}{Q} + 3\right)\right\}. \tag{25}$$

The characteristic equation obtained from (24) by eliminating the constants A_n must be an infinite determinant. However, satisfactory accuracy can be obtained in the calculation of the lowest eigenvalue λ by equating the leading term to zero. The full-line curve of figure 2 has been calculated in this way. The curve is qualitatively similar to a neutral curve of Rayleigh instability in a porous medium, although the minimum occurs at a lower value of λ and a lower value of a than would be found for horizontal planes spaced a distance κ/W_m apart. When $a = 0{\cdot}4285$, this approximate method gives $\lambda = 6{\cdot}9735$—a result less than $0{\cdot}3\,\%$ higher than the 'exact' value (cf. table 1).

At first, the appearance of a minimum at a finite value of a in the neutral curve for this stability problem would seem to be surprising. By contrast, when a horizontal layer of saturated porous material separates two static liquids which differ in density, it can be shown that the critical Rayleigh number decreases to a minimum value only when the horizontal wave-number of the disturbance tends to zero. A brief heuristic explanation follows. As $z \to -\infty$, a density disturbance of small wave-number a tends to zero approximately as e^z, i.e. the disturbance is confined to a boundary layer of constant thickness (κ/W_m in dimensional units) in which vertical diffusion effects are important and are independent of a. However, the fluid motion associated with the disturbance tends to zero as e^{az}, so that, since a roughly similar flow pattern exists for differing small values of a, the path length must vary as $1/a$. Consequently, the rate of growth of such a disturbance will be proportional to a. It follows that the critical Rayleigh number must tend to infinity as $1/a$ when $a \to 0$, and this is what is actually found.

It is of interest to note that a similar argument, leading to the conclusion that the flow-path length varies as $1/a$, indicates the reason for the linear dependence of ω upon a found by Saffman & Taylor (1958). Here ω is the rate of growth of a disturbance (of wave-number a) to the interface between two immiscible fluids in a porous medium.

At large values of a, the curve of neutral stability in figure 2 becomes parabolic. This occurs because the disturbance is concentrated near the surface of the medium in a region where the density gradient is approximately constant, and

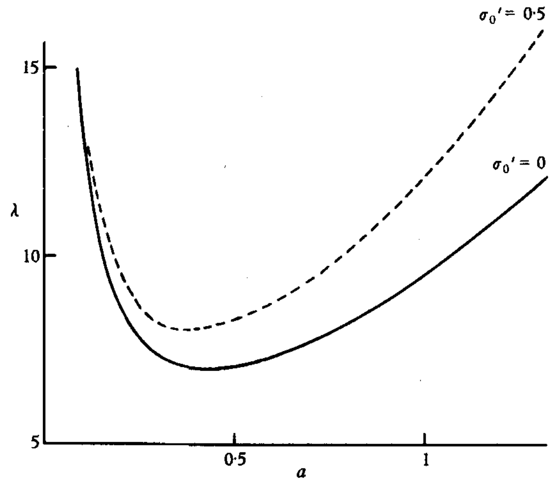

FIGURE 2. Approximate curves showing the value of the Rayleigh number λ for neutral stability plotted as a function of the dimensionless wave-number a. The kinematic viscosity has the functional form
$$\nu = \nu_0(1+\sigma_0' e^z),$$
where σ_0' is assumed constant.

the horizontal dimensions of a typical convection cell are very small compared with the vertical height. Under these conditions, the tendency of the disturbance to grow under the action of vertical forces is independent of the wave-number a, but damping takes place by transverse diffusion, which is proportional to a^2. Similar considerations apply to the motion of a long convection cell in a vertical tube filled with porous material (Wooding 1959).

An extension of the present method has been used to estimate the effects of variations in kinematic viscosity ν which have been neglected in the two previous solutions. For example, if ν is proportional to ρ, one has $\sigma = 1+\sigma_0' e^z$, where $\sigma_0' = \nu_0^{-1}(d\nu/d\rho)_0(\rho_1-\rho_0)$. It is then necessary to solve equation (22) approximately. When $\sigma_0' = 0{\cdot}5$ (a positive value, for which the viscosity increases upwards), the curve of neutral stability is modified approximately as shown by the broken-line curve in figure 2. There is an increase in stability, especially at the higher wave-numbers. An increase in the critical value of the Rayleigh number is found, and the corresponding value of the wave-number is decreased. These effects arise because the fluid motion due to a disturbance of high wave-number is restricted to a layer close to the surface of the medium, where $\nu \approx \nu_0(1+\sigma_0')$, whereas the fluid flow due to a disturbance of low wave-number penetrates more deeply into the medium, where $\nu \approx \nu_0$.

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